A Classical-Modal Interpretation of Smooth Infinitesimal Analysis

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1 Background on SIA

A remarkable development of late 20th C. mathematics, this theory—of "smooth worlds" (as JL Bell describes it)—introduces a nilsquares object, $\Delta =^{df} \{\varepsilon : \varepsilon^2 = 0\}$, along with (modified) axioms of an ordered field along with special axioms governing differentiation and integration of (smooth) functions on a smooth line, *R*. The order axioms are familiar, except that trichotomy is omitted; instead there is the axiom: $0 < a \vee a < 1$, and the axiom $a \neq b \rightarrow a < b \vee b < a$. The central novel axiom is the

Prnciple of Microaffineness: *For any map* $g : \Delta \rightarrow R$ *there exists a unique* b in R *such that for all* ε *in* Δ,

$$g(\varepsilon) = g(0) + b\varepsilon.$$

This is also known as the "Kock-Lawvere axiom". It says that the graph of g is a straight line passing through $(0, g(0))$ with *unique* slope b. Intuitively, this means that any smooth function comes with an infinitesimal tangent vector, called a "linelet", that can be translated or rotated but cannot be "bent". Naturally, the unique b measures the derivative of g at 0. Finally, there is an axiom called the

Constancy Principle: *If $f \colon J \to R$ is such that $f' = 0$ identically, then f is constant.*

This is key in implementing applications of the Fundamental Theorem of the Calculus. (See Bell, pp. 30-31.)

Next, let us motivate the background logic of SIA. It is a consequence of the Microaffineness axiom that not all nilsquares are $= 0$, $i.e.$

$$\neg\forall\varepsilon[\varepsilon^2 = 0 \to \varepsilon = 0].$$

(If $\Delta = \{0\}$, then the b of that axiom would not be unique.) However, if this could be transformed to

$$\exists\varepsilon[\varepsilon^2 = 0 \wedge \varepsilon \neq 0],$$

then by a field axiom we could infer that such an ε has an inverse, ε^{-1}, whence $1 = \varepsilon \cdot \varepsilon^{-1} = \varepsilon^2 \cdot \varepsilon^{-2} = 0$. To block this and related contradictions, the logic of SIA is stipulated to be intuitionist logic, which does not sanction the above quantifier conversion.

2 A Philosophical Challenge re Understanding SIA

In the cases of intuitionist analysis and Bishop's constructive analysis, the restriction to intuitionist logic is readily motivated by the demands for constructive proofs, so e.g. a proof of existence must provide a method of computing or "finding" a suitable witness. It does not suffice to derive a contradiction from the assumption of inexistence. So the classical laws of excluded middle, double negation, and related cannot be part of the logic. Indeed, the whole classical conception of mathematics as a body of mind-independent truths is rejected in favor of a constructive interpretation. The logical operations are all to be understood, not via truth conditions, but via proof conditions, systematized in those known as BHK conditions (for Brouwer-Heyting-Kolmogorov). As a result, there really is no conflict between classical analysis and intuitionist or Bishop analysis *per sē*. There are *apparent* contradictions between them: e.g. intuitionist analysis proves, for arbitrary binary choice sequences r:

$$\neg[\forall r(\forall i(r_i = 0) \vee \exists i(r_i = 1)],$$

which looks like the negation of a classical theorem, but only because the logical symbols are not suitably subscripted to indicate that they are systematically different in meaning from their classical counterparts. We should not even speak of "*the* law of excluded middle", since at least two putative laws are in play. No one from either camp subscribes to the intuitionist would-be law, as it asserts the decidability of all mathematical statements, whereas the classical law is tautological, Carnap-analytic, for the classicist, although the radical intuitionist (e.g. Brouwer or Dummett) claims not even to understand that "law", as it is framed in terms of objective truth conditions. (Of course, as these remarks illustrate, despite lack of genuine logical or mathematical conflicts, there is plenty of conflict on philosophical-foundational issues such as objectivity vs. mind-dependence, meaningfulness of objective claims regarding the infinite, etc.)

Does any of this carry over to SIA in comparison with classical analysis (CA)? We think not. The main reason is that SIA does not even appear to be a constructive theory. It "introduces" nilsquare infinitesimals, but it certainly cannot construct any other than 0. Moreover, the Microaffineness axiom itself does not admit of a constructive interpretation. The constant b asserted to exist, giving the slope of the micro-tangent vector, is not constructed or constructively proved to exist. To be able to do that would be already to have a method of differentiation available, but one of the main points of SIA and this axiom in particular is to provide an *autonomous* method of differentiation as an alternative to the classical limit method! Not surprisingly, SIA's proponents and practitioners never (to our knowledge) claim that their theory or methods are generally constructive.

Indeed, Bell offers arguments that appeal, not to constructivity, but to the demand that all functions treated in SIA be smooth (corresponding to the classical notion of C^{∞}). It would take us too much time to consider these arguments here. One of us has examined them and found them interesting but inconclusive. (See Hellman's "Mathematical Pluralism: the Case of Smooth Infinitesimal Analysis", *JPL* (2006): 621-651)

Here we present what we consider a more successful approach.

The key idea behind our approach was broached but not developed in the Hellman paper just cited. That was that SIA's nilsquares (apart from 0) can be taken as inherently "vague objects", objects lacking determinate identity conditions, where this lack is not due to vagueness of linguistic predicates as standardly invoked to explain failures of bivalence due to borderline cases and absence of linguistic conventions that would resolve such cases. The vagueness of concern regarding SIA can be called "objectual vagueness", a failure to have determinate identity conditions inherent in the objects themselves, not due to imprecision of language. Now, as is well-known, there have been arguments that such objectual vagueness is logically impossible, as in Evans [1978] (see also Lewis [1988]). The Evans argument, however, has been subject to (at least) two fairly recent critiques pointing to fallacious or question-begging steps.) Parsons and Woodruff [1996] provide their own counterexample in a thought experiment. Hellman [2006], *op. cit.,* focuses on what goes wrong with the argument specifically in the context of SIA.

3 A Classical-Modal Interpretation

The first step is to introduce into a classical logical language, with second-order machinery or logic of plurals added to first-order logic with $=$, a *determinateness* operator, D, behaving as a modal operator on formulas. The key idea is to provide axioms governing D providing for cases such that $\neg D(x \neq y)$. Depending on choices of other axioms, we may also have failures of determinateness of positive identites, $\neg D(x = y)$, as well. In the crucial cases of nilsquares ε of SIA, we may have cases such as $\neg D(\varepsilon = 0) \wedge \neg D(\varepsilon \neq 0)$. This corresponds to the fact that, in SIA, we have failures of disjunctions of the form, $\varepsilon = 0 \vee \varepsilon \neq 0$, for $\varepsilon \in \mathbf{\Delta}$. Similarly where '0' is replaced with a variable ranging over $\mathbf{\Delta}$. It is reasonable to endorse

$$DA \rightarrow DDA,$$

for sentences A, which is the characteristic axiom for S4, where D functions as the necessity operator (although with our determinateness interpretation, suitable for use in connection with SIA). "Possibly", P, then is defined as $\neg D \neg$. (This can be read, "is not determinately ruled out". The other axioms of S4 are evident as well for "It is determinate that" as the $\Box$.

The next step is to apply the well-known translation, due to Gödel, of a given theory based on intuitionistic logic into the language of S4. Here are the clauses, where A* is the Gödel translate of SIA formula A:

- If A is atomic, A* is DA;

- $(\neg A)$* is $D(\neg A$*$)$;

- $(A \vee B)$* is A*$\vee B$*; likewise * commutes with $\wedge$;

- $(A \rightarrow B)$* is $D(A$*$\rightarrow B$*$)$;

- $(\forall x A)$* is $D(\forall x A$*$)$; likewise for the existential quantifier.

In particular, $(x = y)$* is $D(x = y)$; and $(x \neq y)$* is $D\neg D(x = y)$, as motivated above in the case of "non-0 nilsquares".

Now we stipulate that the *-translates of the SIA axioms (as presented in Bell [1998], Ch. 8) are to be taken as axioms of our S4 classical-modal theory, CM.

Then we can apply the key theorem of Gödel's governing his *-translation, namely that this translation is proof-theoretcally faithful:

$$A_1, ..., A_n \vdash_{SIA} B \text{ iff } A_1^*, ..., A_n^* \vdash_{CM} B^*.$$

(Mirroring Theorem)

This of course guarantees that if SIA is consistent, then so is CM (as we have so far defined it). Furthermore, since the underlying logic of S4 here is classical, CM appears to be exactly what we've been seeking, a classical-modal interpretation of SIA.

Let us now observe what happens when we apply this to the initially puzzling SIA result that, although

$$\vdash_{SIA} \neg\forall x[x^2 = 0 \rightarrow x = 0],$$

nevertheless, if SIA is consistent, then SIA *cannot* prove

$$\exists x[x^2 = 0 \wedge x \neq 0].$$

Applying the Mirroring Theorem to the above SIA theorem, we have

$$\vdash_{CM} D\neg D\forall x D[D(x^2 = 0) \rightarrow D(x = 0)],$$

whence, by a quantifier conversion,

$$\vdash_{CM} D\neg D\neg\exists x\neg D[D(x^2 = 0) \to D(x = 0)],$$

equivalently, in terms of possibility,

$$\vdash_{CM} DP\exists xP[D(x^2 = 0) \wedge \neg D(x = 0)].$$

Thus we have derived in CM the *possible existence* of something possibly a nilsquare not determinately $= 0$. (Of course, this isn't expressible in SIA, which lacks modal operators. It is only equivalent *in CM* to the CM-translate of the SIA theorem displayed above.) This gives substance to Bell's remark that non-zero nilsquares exist only in a "potential sense", as their outright existence cannot be proved or even consistently added to SIA, as already observed above, but nevertheless SIA survives in that they "cannot be ruled out". (See Bell [1998], e.g. p. 7, along with his citation there of Aristotle and Bradwardine as having long ago expressed the idea that infinitesimal quantities have only "a potential existence".)

Can the outright existence, in addition to possible existence, of non-determinately $= 0$ nilsquares be proved in CM? We haven't seen how. Nevertheless, it can be shown that the existence of non-$D = 0$ nilsquares can be consistently added to our CM theory, provided that the latter itself is consistent (which it is if SIA is—and we note that topos models of SIA along with synthetic differential geometry have been derived). Indeed, suppose CM$^+$ were inconsistent, where CM$^+$ is the result of adding to CM as a new axiom,

$$\exists x[D(x^2 = 0) \wedge \neg D(x = 0)].$$

Then CM would prove the negation of this, so we would also have, by necessitation,

$$\vdash_{CM} D\forall x[D(x^2 = 0) \rightarrow D(x = 0)].$$

Then by the converse Barcan formula (valid in S4) and necessitation, we have

$$\vdash_{CM} D\forall x D[D(x^2 = 0) \rightarrow D(x = 0)].$$

But the negation of this is implied by the Mirroring Theorem, as it follows directly, by D-elimination, from

$$\vdash_{CM} D\neg D\forall x D[D(x^2 = 0) \rightarrow D(x = 0)],$$

the result of applying Mirroring to

$$\vdash_{SIA} \neg\forall x[x^2 = 0) \rightarrow x = 0].$$

This establishes that if CM itself is consistent, so is CM$^+$, as claimed.■

Remark on Meaning of Logical Terms: Our CM interpretation of SIA is silent on the question of meanings of SIA's logical terms (including '$=$'). But in the cases of $x = y$ and $x \neq y$, the CM translates must be $D(x = y)$ and $D\neg D(x = y)$, respectively, and it is reasoable to maintain that determinateness is built into the classical understanding of $=$ and $\neq$. Again, this is marked contrast to constructivist mathematics, which adopts a constructive interpretation of identities and non-identites. Furthermore, our strategy for accounting for the failure in SIA of classical laws such as excluded middle, double negation elimination, trichotomy among reals, etc. is based on indeterminacy of identity conditions among nil-squares rather than "change of meaning" of logical terms. This is not to deny that there may also be differences of meaning of SIA's logical terms in comparison with those of classical mathematics, but that matter is independent of our present approach.

Finally on this matter, one positive aspect of not positing such meaning differences is that SIA theorems, such as the fundamental theorem of the calculus (restricted to smooth functions), can be understood as providing alternative proofs of *the same* respective classical theorems rather than of theorems that merely appear the same. (Cf. Shapiro [2014], pp. 108-109.)

4 Improving on CM

There is, however, one serious drawback of the CM theory as developed above. That is that, due to the bivalence of the underlying classical logic, every world w of a Kripke frame for our CM theory satisfies either $e = 0$ or $e \neq$

0, for nilsquare e even if it also satisfies $\neg D(e = 0)$ and $\neg D(e \neq 0)$. This thwarts our aim of satisfying the possible existence of a nilsquare e satisfying $\neg D(e = 0)$ without satisfying $e \neq 0$ *simpliciter.* For that, CM must be modified.

One way to do that is to move to an otherwise classical 3-valued logic, say Kleene's strong version. Call our S4 modal language and theory, based on this 3-valued logic, 'SCM' (for "semi-classical modal"). Although the classical quantifier conversion laws are respected, sentences such as $e = 0$, $e \neq 0$, etc., can be assigned the intermediate truth-value, call it 'I' for "indefinite". The semantical rules governing D-sentences at worlds of a Kripke frame will then read:

- $w \models D\varphi$ iff $\forall w'[w'$ accessible from $w \rightarrow w' \models \varphi]$;

- $w \models \neg D\varphi$ iff *either* $\exists w'[w'$ accessible from $w \wedge w' \models \neg\varphi]$ *or* $\exists w'[w'$ accessible from $w \wedge w' \models \varphi$ is $I]$

As a result, any sentence of our SCM language beginning with a 'D' will be evaluated as either true or false at any world.

But now consider the Gödel-translate S* of an arbitrary SIA sentence, S. Note that S* will either begin with a 'D' operator, or its truth-value will be determined by sentences beginning with a 'D' operator via finitely many steps of forming truth-functional compounds. The result is that all translates of SIA sentences will satisfy classical bivalence, never being evaluated 'I'. Thus we have the mirroring theorem, now relating provability in SIA with provability in our SCM theory, inherited directly from the original mirroring theorem relating SIA and CM. SCM is thus also proof-theoretically faithful with respect to SIA via the Gödel translation. But SCM brings with it the freedom to satisfy, e.g. $\neg D(e = 0)$ and $\neg D(e \neq 0)$ while still not satisfying either $e = 0$ or $e \neq 0$, assigning them both I, as desired.

5 Further Axioms

The transition from CM to SCM brings with it some interesting differences regarding what further axioms it would be acceptable to add to the theories. Consider, for example, the "stability of identity":

$$x = y \to D(x = y). \tag{SI}$$

This can be derived in CM using the Leibnizian law of "non-identity of discernibles" in an Evans style argument. It is also arguably part of the classical conception of identity, and most modal systems endorse it. Furthermore, below, we will expose an unattractive consequence of giving it up in favor of an alternative stability principle we consider. (One of us, SS, favors (SI). The other, GH, leans toward the alternative.) What about adding (SI) to SCM? There it would imply that any world (of a Kripke frame) that satisfied $\neg D(e = 0)$, for a nilsquare e, would also satisfy $e \neq 0$, contrary to one motivation of SCM, to be able to assign I (indefinite) rather than T or F to sentences like '$e \neq 0$' and '$e = 0$', especially in such circumstances. Further, consider the Evans argument itself, using that same Leibnizian law or a similar lambda abstraction: That begins with a hypothetical premise, in the present case, $\neg D(e = 0)$, then invoking $D(0 = 0)$, then concluding $e \neq 0$ *simpliciter.* Thus, while the Leibnizian law can be adopted in CM, it cannot in SCM in full generality, if the above consequence of (SI) is to be avoided, for then the Evans argument has to be blocked.

Can the proponent of (SI) simply add it to SCM consistently? Provided CM with the Leibnizian law is consistent, then trivially, yes, as a model of CM is just a special case of a model of SCM in which no sentence is evaluated as I in any world, and (SI) is derivable in CM given the Leibnizian law.

Here is the alternative to (SI) we consider: It's the *determinateness (stability) of indeterminateness of identity* among nilsquares, e.g. in the form,

$$\neg D(e = 0) \rightarrow D\neg D(e = 0). \qquad \text{(S}\neg\text{DI)}$$

Thus, if a world w satisfies '$\neg D(e = 0)$', for a nilsquare e, then so does any world w' accessible from w. The intuition behind this is that the "objectual vagueness" of a nilsquare e is an essential property of it, something not expressible in SIA but expressible in our CM and SCM languages. Thus, even if w' satisfies '$e = 0$'—and some world w' accessible from w, satisfying '$\neg D(e = 0)$', must to avoid w's satisfying $D(e \neq 0)$, which leads to contradiction as we've seen—still in some further w'' accessible from w' and therefore from w, $w'' \models e \neq 0$, i.e. e "splits" from 0, contrary to ordinary intuitions about identity and contrary to (SI) above. Nevertheless, we think we can claim to provide a Kripke modal model of SCM + (S¬DI). The key points are as follows:

(i) Begin with a model of SCM (guaranteed by the completeness theorem for S4 together with the consistency of SCM via the mirroring theorem, assuming consistency of SIA itself).

(ii) As $\vdash_{SIA} \neg\forall e[e^2 = 0 \rightarrow e = 0]$, by mirroring (and D-elimination), we have:

$$\vdash_{SCM} P\exists e[e^2 = 0 \wedge \neg D(e = 0). \qquad (P\neg D)$$

So given any world w of the initial model, some world w' accessible from w satisfies $\neg D(e = 0)$ for a nilsquare e. Then, if there isn't already a w'' accessible from w' such that $w'' \models e \neq 0$, add such a world, stipulating that it also satisfies any sentence with an initial D satisfied in any world from which w'' is accessible. Also, since the *determinateness* of $P\neg D$ is directly what is guaranteed by mirroring, apply this latter procedure to all worlds.

(iii) All worlds must satisfy $\neg D(e \neq 0)$ for any nilsquare e. Thus from any world w there must be an accessible w' such that $w' \models e = 0$. Such a world is then to be added as accessible to any world added by (ii) from which there is not already such an accessible w'.

(iv) Finally, take the minimal closure of the set of the worlds of the initial SCM model under the operations given in (ii) and (iii). This should provide our desired model of SCM + (S¬DI).

Final remark: The main philosophical disadvantage of the axiom (S¬DI) is the need for the counterintuitive "splitting" of non-zero nilsquares away from 0 while preserving their original identities. (This suggests that we shouldn't think of a sentence of the form $e = 0$ as a true identity after all, but perhaps as a weaker kind of indiscernibility.) Fusion and splitting themselves are not particularly problematic; we're all familiar with examples from e.g. cell biology. But the result of fusion is normally loss of earler identities, i.e. creation of a new synthesis, from which any future fission gives rise to multiple new entities (or at least some new ones—one could persist as the "mother" of some new ones). So the phenomenon arising from SCM + (S¬DI) is certainly cause for wonder. But then, so is the existence of SIA itself!

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